$$\rho(v) = \exp\left\{\frac{\gamma}{2}\left(1 - \frac{1}{v}\right)\right\}$$

ψ, Δ = constants defined by Equation (33)
Ω, Λ = constants defined by Equation (37)

Subscripts

c = critical value
f = bulk value
h = heat transfer
m = mass transfer
n = n-th term in series
$*$ = lower bifurcation value

Superscripts

$*$ = upper bifurcation value

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Multivariable Controller Design for Linear Systems Having Multiple Time Delays

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A multivariable, multidelay compensator, capable of handling general, linear time delay problems is derived in a form applicable both in continuous and discrete time. The controller developed is shown to have the same structure as the linear-quadratic optimal feedback controller for input delays and reduces to the Smith predictor (and the analytical predictor) for the special case of a single time delay. Some examples representative of engineering practice are used to demonstrate the effectiveness of the controller.

SCOPE

Time delays in feedback control loops often are a serious obstacle to good process operation. Such delays prevent high controller gains from being used, leading to offset and sluggish system response. Smith (1957, 59) suggested a compensator design which effectively removes a single delay from the feedback loop. This result has been extended to discrete time operation and multivariable systems having a single delay (e.g., Moore et al. 1970, Alevisakis and Seborg 1973, 1974), but the common situation of multiple time delays has remained a problem. In the present work, a new multivariable, multidelay compensator is developed to allow mitigation of the effects of multiple delays in multivariable control problems.

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CONCLUSIONS AND SIGNIFICANCE

The new multidelay compensator is capable of handling any combination of state delays, control delays and output delays. For input delays, the resulting controller structure is shown to be the same as for the linear quadratic optimal feedback controller (Soliman and Ray 1972) developed earlier. Simulation testing of the multidelay compensator on chemical reactor and distillation column examples shows definite improvements over the performance of other controller designs. The on-line implementation requirements would appear to be quite modest.

There is a great variety of processes whose dynamics can be adequately represented by multivariable transfer functions having time delays. Some of the more important processes in this category are 1) distillation columns, (cf Moczek et al. (1965), Kim and Freidly (1974), Wood and Berry (1973)); 2) extraction and adsorption processes (e.g., Biery and Boylan (1962) and Horner and Schiesser (1965)); and 3) heat exchangers, (e.g., Stewart et al. (1961)). The general form of the transfer function matrix between outputs, **y** and controls, **u**, is

$$\bar{\mathbf{y}}(s) = \mathbf{G}(s)\,\bar{\mathbf{u}}(s) \tag{1}$$

where **y** is an l vector of outputs and **u** is an m vector of controls. Similarly, the transfer function between the outputs and disturbances, **d** is

$$\bar{\mathbf{y}}(s) = \mathbf{G}_d(s)\,\bar{\mathbf{d}}(s) \tag{2}$$

where **d** is a k vector of disturbances. The transfer function matrices, $\mathbf{G}(s)$, $\mathbf{G}_d(s)$ are of the form

$$\mathbf{G}(s) = \begin{bmatrix} g_{11} & g_{12} & \cdots & g_{1m} \\ g_{21} & \ddots & & \\ \vdots & & \ddots & \\ g_{l1} & \cdots & \cdots & g_{lm} \end{bmatrix};$$

$$\mathbf{G}_d(s) = \begin{bmatrix} g^d_{11} & g^d_{12} & \cdots & g^d_{1k} \\ g^d_{21} & & & \\ \vdots & & & \\ g^d_{l1} & \cdots & \cdots & g^d_{lk} \end{bmatrix} \tag{3}$$

In many practical applications where the transfer functions are fitted to experimental data, $g_{ij}(s)$ or $g^d{}_{ij}(s)$ have the rather simple form

$$g_{ij}(s) = \frac{k_{ij} \prod_{p=1}^{n} (h_{ijp}s + 1)e^{-\alpha_{ij}s}}{\prod_{q=1}^{r} (f_{ijq}s + 1)} \tag{4}$$

where

r = order of the transfer function
n = number of zeroes ($n < r$)
$-(h_{ijp})^{-1}$ = zeroes of the transfer function g_{ij}
$-(f_{ijq})^{-1}$ = poles of the transfer function g_{ij}
k_{ij} = steady state gain of g_{ij}
α_{ij} = time delay associated with g_{ij}

However, more complex transfer functions sometimes arise, as illustrated in the next section.

A block diagram for the system under conventional feedback control may be seen in Figure 1, where $\mathbf{G}_c$ represents the feedback controller, **H** the output measurement device, and $\mathbf{y}_d$ the output set-point. The closed loop response for the conventional controller then may be given by

$$\bar{y}(s) = (\mathbf{I} + \mathbf{G}\mathbf{G}_c\mathbf{H})^{-1}[\mathbf{G}\mathbf{G}_c\bar{\mathbf{y}}_d(s) + \mathbf{G}_d\bar{\mathbf{d}}(s)] \tag{5}$$

In the absence of time delays, there are many multivariable control design procedures available (cf. Ray, in press) for choosing the elements $\mathbf{G}_c$ in order to achieve good control system performance. In addition, for multivariable systems having only one time delay, useful design methods are also available (Alevisakis and Seborg 1973, 1974). However, when there are multiple delays in the transfer function as in Equation (4), the choice of design algorithms is more limited. For example, optimal feedback control methods (cf. Soliman and Ray 1972*a*, *b*) and Inverse Nyquist Array Techniques (Rosenbrock 1974) have been used. While these procedures are straightforward and have been shown to work well, the presence

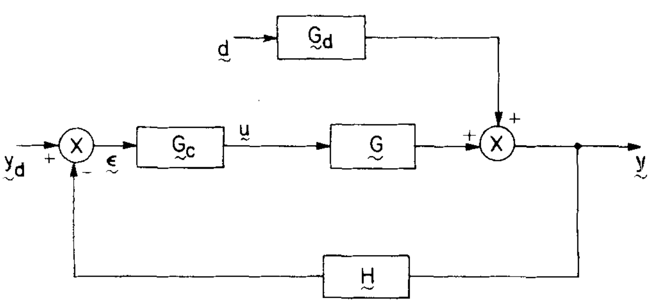

Figure 1. Block diagram for conventional feedback control of a multivariable system.

of time delays requires rather extensive computational effort to determine the controller structure for $\mathbf{G}_c$. Thus it would be advantageous to have simpler design procedures for multivariable systems having multiple time delays.

The purpose of this article is to present new time delay compensation techniques which largely eliminate the effects of time delays and allow conventional multivariable controller design procedures to be used for systems with multiple time delays. The design procedure is formulated so that it may be applied in either a continuous time or a discrete time mode. In the next section, we discuss the various forms time delay models may take and show how all of these can be handled by our design procedure. Then we discuss earlier single variable and single delay formulations and indicate how they become special cases of our more general approach. Finally, we present some examples to illustrate the controller design procedure and the resulting control system performance.

TIME DELAY SYSTEM FORMULATIONS AND REALIZATIONS

There are many different forms that linear systems with time delays may take. The most general formulation in the time domain is

$$\dot{\mathbf{x}} = \sum_i \mathbf{A}_i\mathbf{x}(t-\rho_i) + \sum_j \mathbf{B}_j\mathbf{u}(t-\beta_j) + \sum_k \mathbf{D}_k\mathbf{d}(t-\delta_k) \tag{6}$$

$$\mathbf{y} = \sum_i \mathbf{C}_i\mathbf{x}(t-\gamma_i) + \sum_j \mathbf{E}_j\mathbf{u}(t-\epsilon_j) \tag{7}$$

where $\mathbf{x}$ is an n vector of state variables and the ρ_i, β_j, δ_k, γ_i, ϵ_j are time delays. By taking Laplace transforms of (6) and (7), transfer function matrices of the form

$$\overline{\mathbf{y}}(s) = \mathbf{G}(s)\overline{\mathbf{u}}(s) + \mathbf{G}_d(s)\overline{\mathbf{d}}(s) \tag{8}$$

arise, where now

$$\mathbf{G}(s) = \sum_j \mathbf{E}_j e^{-\epsilon_j s} + \{[\sum_i \mathbf{C}_i e^{-\gamma_i s}][s\mathbf{I} - \sum_i \mathbf{A}_i e^{-\rho_i s}]^{-1}\}[\sum_j \mathbf{B}_j e^{-\beta_j s}] \tag{9}$$

$$\mathbf{G}_d(s) = \{[\sum_i \mathbf{C}_i e^{-\gamma_i s}][s\mathbf{I} - \sum_i \mathbf{A}_i e^{-\rho_i s}]^{-1}\}[\sum_k \mathbf{D}_k e^{-\delta_k s}] \tag{10}$$

Note that $\mathbf{G}(s)$, and $\mathbf{G}_d(s)$ in Equations (9) and (10) are very much more general than the more commonly encountered forms of $\mathbf{G}(s)$ and $\mathbf{G}_d(s)$ given by Equation (4). However in normal engineering practice, the usual modelling procedure is to carry out step, pulse, or frequency response measurements on the actual process to obtain an approximate transfer function model in the form of Equations (3, 4). The more complex forms (9, 10) usually arise when the model is formulated as differential equations and transformed to the Laplace domain.

Through the modification of classical realization theory (cf. Ogunnaike, in press) it is possible to convert any transfer function with time delays to a time domain representation in the form (6, 7). As in the case of systems without delays there can be many such realizations corresponding to a given transfer function pair $\mathbf{G}(s)$, $\mathbf{G}_d(s)$. However, it is usually best to choose a minimal, cannonical form. Examples given below illustrate this point.

Discrete Formulations

The most general stationary state-space model for discrete-time systems with multiple time delays is

$$\mathbf{x}(n+1) = \sum_i \boldsymbol{\Phi}_i\mathbf{x}(n-\rho_i) + \sum_j \boldsymbol{\Delta}_j\mathbf{u}(n-\beta_j) + \sum_k \boldsymbol{\Theta}_k\mathbf{d}(n-\delta_k) \tag{11}$$

$$\mathbf{y}(n) = \sum_i \mathbf{C}_i\mathbf{x}(n-\gamma_i) + \sum_j \mathbf{E}_j\mathbf{u}(n-\epsilon_j) \tag{12}$$

where $\boldsymbol{\Phi}_i$, $\boldsymbol{\Delta}_j$, $\boldsymbol{\Theta}_k$, $\mathbf{C}_i$, $\mathbf{E}_j$ are constant matrices of appropriate dimensions.

Taking the z transforms of (11) and (12) and using the notation $\hat{\mathbf{x}}(z)$ to denote the z-transform of $\mathbf{x}(n)$ we have

$$z\hat{\mathbf{x}}(z) = \sum_i \boldsymbol{\Phi}_i z^{-\rho_i}\hat{\mathbf{x}}(z) + \sum_j \boldsymbol{\Delta}_j z^{-\beta_j}\hat{\mathbf{u}}(z) + \sum_k \boldsymbol{\Theta}_k z^{-\delta_k}\hat{\mathbf{d}}(z) \tag{13}$$

$$\hat{\mathbf{y}}(z) = \sum_i \mathbf{C}_i z^{-\gamma_i}\hat{\mathbf{x}}(z) + \sum_j \mathbf{E}_j z^{-\epsilon_j}\hat{\mathbf{u}}(z) \tag{14}$$

It is usual in less complicated formulations, to rearrange (13) and (14) in a form relating $\hat{\mathbf{y}}(z)$ to $\hat{\mathbf{u}}(z)$ and $\hat{\mathbf{d}}(z)$. However, because of the generality of these equations, we find it more beneficial at this stage to introduce the matrices $\hat{\mathbf{Q}}(z)$, $\hat{\mathbf{Q}}_d(z)$, as matrix "transfer functions" between the z-transform variables $\hat{\mathbf{y}}(z)$, $\hat{\mathbf{u}}(z)$ and $\hat{\mathbf{y}}(z)$, $\hat{\mathbf{d}}(z)$ respectively. Observe that this definition implies that $\hat{\mathbf{Q}}(z)$ and $\hat{\mathbf{Q}}_d(z)$ are 'discrete' versions of $\mathbf{G}(s)$, $\mathbf{G}_d(s)$ of Equation (8). Hence we have

$$\hat{\mathbf{y}}(z) = \hat{\mathbf{Q}}(z)\hat{\mathbf{u}}(z) + \hat{\mathbf{Q}}_d(z)\hat{\mathbf{d}}(z) \tag{15}$$

where, from (13) and (14) we note that

$$\hat{\mathbf{Q}}(z) = \sum_j \mathbf{E}_j z^{-\epsilon_j} + \{[\sum_i \mathbf{C}_i z^{-\gamma_i}][z\mathbf{I} - \sum_i \boldsymbol{\Phi}_i z^{-\rho_i}]^{-1}\}[\sum_j \boldsymbol{\Delta}_j z^{-\beta_j}] \tag{16}$$

$$\hat{\mathbf{Q}}_d(z) = \{[\sum_i \mathbf{C}_i z^{-\gamma_i}][z\mathbf{I} - \sum_i \boldsymbol{\Phi}_i z^{-\rho_i}]^{-1}\}[\sum_k \boldsymbol{\Theta}_k z^{-\delta_k}] \tag{17}$$

TIME DELAY COMPENSATION

In the late 1950s, Smith (1957, 1959) developed a time delay compensator for a single delay in a single control loop which eliminated the delay from the feedback loop, allowing higher controller gains. This compensator, termed the Smith Predictor, is shown in the block diagram in Figure 2. Moore et al. (1970), working with a scalar state-space model, used the analytic solu-

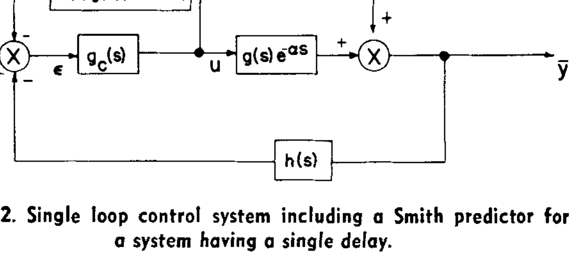

Figure 2. Single loop control system including a Smith predictor for a system having a single delay.

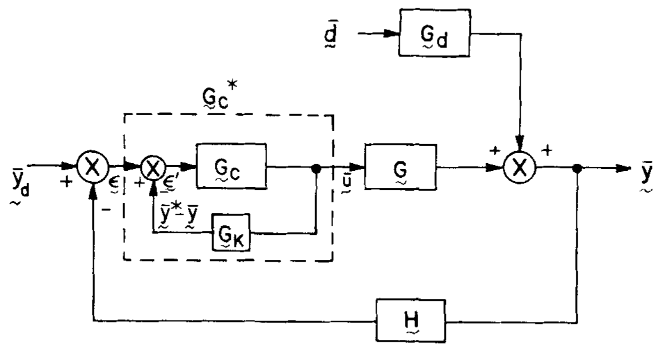

Figure 3. Block diagram of feedback control of multivariable system with time delay compensation.

tion of the modelling equation to predict the value of the state one delay time ahead. This analytical predictor was developed primarily for sampled data systems and hence included in its structure corrections for the effect of sampling and the zero-order hold. Alevisakis and Seborg (1973, 1974) have expanded both the discrete time and continuous time results to multivariable systems having a single delay in either the output or control variables. In each of these earlier algorithms involving only a single time delay, the predictor estimates the output variable a "time delay" in the future, *or equivalently* eliminates the delay from the output variable signal sent to the controller.

For the more general case of multivariable systems with multiple delays in the transfer function matrices, $\mathbf{G}$, $\mathbf{G}_d$, $\mathbf{H}$, it is possible to design a compensator which eliminates the time delays in the output variables sent to the controller. This is *not equivalent* to predicting the output variable at some single time in the future, but corresponds to the prediction of certain state variables at various specific times in the future. Before proceeding further with this interpretation, let us derive the multivariable, multiple delay analog of the Smith predictor.

By analogy with the philosophy of the original Smith predictor, the corresponding multivariable multidelay compensator would have the structure shown in Figure 3 where the compensator $\mathbf{G}_K$ could have many forms. We demonstrate below that with the particular choice

$$\mathbf{G}_K = \mathbf{H}^* \mathbf{G}^* - \mathbf{HG} \tag{18}$$

(where $\mathbf{H}^*$, $\mathbf{G}^*$ are the transfer function matrices $\mathbf{H}$, $\mathbf{G}$ without the delays) the compensator eliminates the delays in the output variable signal sent to the controller. To see this, let us evaluate the inner loop $\mathbf{G}_c^*$, in Figure 3. Here

$$\bar{\mathbf{u}} = \mathbf{G}_c^* \,\boldsymbol{\varepsilon} \tag{19}$$

or

$$\mathbf{G}_c^* = (\mathbf{I} + \mathbf{G}_c\mathbf{G}_K)^{-1}\,\mathbf{G}_c \tag{20}$$

Thus the entire transfer function is

$$\bar{\mathbf{y}} = (\mathbf{I} + \mathbf{G}\mathbf{G}_c^*\mathbf{H})^{-1}\,[\mathbf{G}\mathbf{G}_c^*\bar{\mathbf{y}}_d + \mathbf{G}_d\bar{\mathbf{d}}\,] \tag{21}$$

One of the principal goals of time delay compensation is to eliminate the time delay from the characteristic equation of the closed loop transfer function so that higher controller gains and standard multivariable controller design algorithms may be used. The multidelay compensator noted above achieves this goal. Substituting Equations (18, 20) into (21) yields

$$\bar{\mathbf{y}} = (\mathbf{I} + \mathbf{G}\mathbf{R}^{-1}\,\mathbf{G}_c\mathbf{H})^{-1}\,[\mathbf{G}\mathbf{R}^{-1}\,\mathbf{G}_c\bar{\mathbf{y}}_d + \mathbf{G}_d\bar{\mathbf{d}}\,] \tag{22}$$

where

$$\mathbf{R} = \mathbf{I} + \mathbf{G}_c\,(\mathbf{H}^*\mathbf{G}^* - \mathbf{HG}) \tag{23}$$

Now it is easy to show that if $\mathbf{G}$ is square* and nonsingular then the following identity holds.

$$(\mathbf{I} + \mathbf{G}\mathbf{R}^{-1}\,\mathbf{G}_c\mathbf{H})^{-1} = \mathbf{G}(\mathbf{R} + \mathbf{G}_c\mathbf{HG})^{-1}\,\mathbf{R}\mathbf{G}^{-1} \tag{24}$$

Now, from (23),

$$(\mathbf{R} + \mathbf{G}_c\mathbf{HG}) = \mathbf{I} + \mathbf{G}_c\mathbf{H}^*\mathbf{G}^* \tag{25}$$

Thus (22) becomes

$$\bar{\mathbf{y}} = \mathbf{G}(\mathbf{I} + \mathbf{G}_c\mathbf{H}^*\mathbf{G}^*)^{-1}\,\mathbf{G}_c\bar{\mathbf{y}}_d + \mathbf{G}(\mathbf{I} + \mathbf{G}_c\mathbf{H}^*\mathbf{G}^*)^{-1}\,\mathbf{R}\mathbf{G}^{-1}\,\mathbf{G}_d\bar{\mathbf{d}} \tag{26}$$

As shown in the Appendix A, the stability of the closed loop system including the compensator (Figure 3) is determined by the characteristic equation

$$|\mathbf{I} + \mathbf{G}_c\mathbf{H}^*\mathbf{G}^*| = 0 \tag{27}$$

Thus the compensator has, indeed, removed the time delays from the characteristic equation and the delays do not influence the closed loop stability, if the model matches the plant exactly. In practice, modelling errors usually allow some delays to remain in the system so that one should be conservative in controller tuning. Even with very conservative gains chosen for the compensated system, the control system will be much better than with no compensation.

It is useful to note that if the compensator is chosen in the form of Equation (18), then the Smith predictor, the analytical predictor, and the Alevisakis and Seborg predictors all become special cases of our general multidelay compensator. In addition, as shown in Appendix B, the analytical predictor is equivalent to a discrete time formulation of the original Smith predictor for a first order plus delay system.

It is important to realize that any type of delay can be dealt with through the use of this compensator, even the very complex types shown in Equations (9, 10). Some examples will follow.

Physical interpretation of the effective action of this compensator is useful and can be illustrated by the following 2 x 2 example system with $\mathbf{H} = \mathbf{H}^* = \mathbf{I}$ and

$$\mathbf{G} = \begin{bmatrix} a_{11}(s)e^{-\alpha_{11}s} & a_{12}(s)e^{-\alpha_{12}s} \\ a_{21}(s)e^{-\alpha_{21}s} & a_{22}(s)e^{-\alpha_{22}s} \end{bmatrix} \tag{28}$$

By definition

$$\mathbf{G}^* = \begin{bmatrix} a_{11}(s) & a_{12}(s) \\ a_{21}(s) & a_{22}(s) \end{bmatrix} \tag{29}$$

and for $\mathbf{G}_c$ consisting of two proportional controllers

$$\mathbf{G}_c = \begin{bmatrix} k_{11c} & 0 \\ 0 & k_{22c} \end{bmatrix}$$

then

$$\mathbf{I} + \mathbf{G}_c\mathbf{H}^*\mathbf{G}^* = \begin{bmatrix} 1 + k_{11c}a_{11} & k_{11c}a_{12} \\ k_{22c}a_{21} & 1 + k_{22c}a_{22} \end{bmatrix} \tag{30}$$

and the characteristic equation is

$$|\mathbf{I} + \mathbf{G}_c\mathbf{H}^*\mathbf{G}^*| = 1 + k_{11c}a_{11} + k_{22c}a_{22} + k_{11c}k_{22c}\,[a_{11}a_{22} - a_{12}a_{21}] = 0,$$

which contains no time delays.

* This is by no means restrictive, since we can always construct pseudo-inverses or generalized inverses of the matrix G. We however note that in most cases, the number of inputs and outputs used in a feedback control scheme are equal, hence G is square.

Further, because of the compensator (cf., Figure 3), the error signal fed to the controller is

$$\boldsymbol{\varepsilon}' = \overline{\mathbf{y}}_d - \overline{\mathbf{y}}^* \tag{31}$$

Here

$$\overline{\mathbf{y}}^* = \mathbf{G}^*\overline{\mathbf{u}} \tag{32}$$

is the output variable without delays in **G**. It is interesting to note that $\overline{\mathbf{y}}^*$ does not correspond to the actual value of $\overline{\mathbf{y}}$ at any specific time, but is a totally fictitious value composed of certain predicted "state" variables. To illustrate let us define variables x_{ij} by

$$\overline{x}_{ij}(s) = a_{ij}(s)e^{-\alpha_{ij}s}\,\overline{u}_j(s) \tag{33}$$

Thus the system with delays

$$\overline{\mathbf{y}}(s) = \mathbf{G}(s)\,\overline{\mathbf{u}}(s) \tag{34}$$

may be written

$$\left.\begin{aligned} \overline{y}_1(s) &= \overline{x}_{11}(s) + \overline{x}_{12}(s) \\ \overline{y}_2(s) &= \overline{x}_{21}(s) + \overline{x}_{22}(s) \end{aligned}\right\} \tag{35}$$

or in the time domain

$$\left.\begin{aligned} y_1(t) &= x_{11}(t) + x_{12}(t) \\ y_2(t) &= x_{21}(t) + x_{22}(t) \end{aligned}\right\} \tag{36}$$

By adding the compensator to the loop, the controller receives $\overline{y}^*(s)$ defined by Equation (32) which may be written as

$$\left.\begin{aligned} \overline{y}_1^*(s) &= e^{\alpha_{11}s}\,\overline{x}_{11}(s) + e^{\alpha_{12}s}\,\overline{x}_{12}(s) \\ \overline{y}_2^*(s) &= e^{\alpha_{21}s}\,\overline{x}_{21}(s) + e^{\alpha_{22}s}\,\overline{x}_{22}(s) \end{aligned}\right\} \tag{37}$$

where $\overline{x}_{ij}(s)$ is defined by (33). In the time domain

$$\left.\begin{aligned} y_1^*(t) &= x_{11}(t + \alpha_{11}) + x_{12}(t + \alpha_{12}) \\ y_2^*(t) &= x_{21}(t + \alpha_{21}) + x_{22}(t + \alpha_{22}) \end{aligned}\right\} \tag{38}$$

and the compensated output $\mathbf{y}^*(t)$ is composed of predictions of the "state" variables x_{ij}. Because the time delays in all the "state" variables are different, $\mathbf{y}^*(t)$ is a totally fictitious output never attained in reality. However, if the control system is stable, then $\mathbf{y}^* \to \mathbf{y}$ as $t \to \infty$ and the fictitious value $\mathbf{y}^*$ is a good "aiming point" for the controller. The actual performance of the compensator shall be illustrated in the examples to follow. It is interesting to note (cf., Appendix C) that the feedback law resulting from using the multidelay compensator has exactly the same structure as that obtained from the optimal feedback controller (Soliman and Ray 1972), and therefore, can be made an optimal controller with proper tuning.

DISCRETE FORMULATION OF THE COMPENSATOR

Following from the definition of $\hat{\mathbf{Q}}(z)$ and $\hat{\mathbf{Q}}_d(z)$ in Equations (15, 16, 17) the discrete system can be represented in terms of the z-transforms equation

$$\hat{\mathbf{y}}(z) = \hat{\mathbf{Q}}(z)\,\hat{\mathbf{u}}(z) + \hat{\mathbf{Q}}_d(z)\,\hat{\mathbf{d}}(z) \tag{39}$$

with the multidelay compensator, $\hat{\mathbf{u}}(z)$ will be defined as

$$\hat{\mathbf{u}}(z) = \hat{\mathbf{G}}_c^*\,\hat{\boldsymbol{\varepsilon}}(z)$$

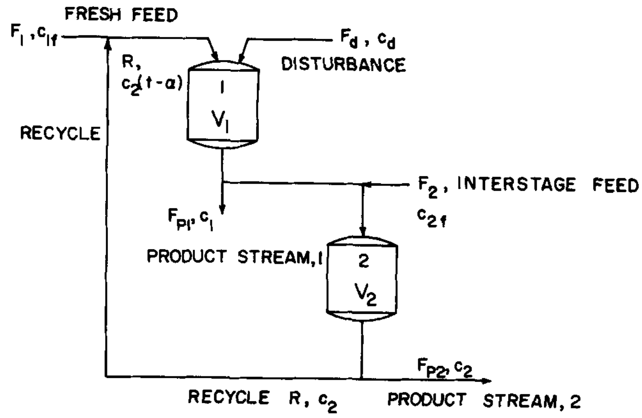

Figure 4. Two stage chemical reactor train with delayed recycle.

where, with the compensator block $\hat{\mathbf{G}}_K$ now given by

$$\hat{\mathbf{G}}_K = \hat{\mathbf{H}}^*\hat{\mathbf{Q}}^* - \hat{\mathbf{H}}\,\hat{\mathbf{Q}} \tag{40}$$

$\hat{\mathbf{G}}_c^*$ is given by

$$\hat{\mathbf{G}}_c^* = (\mathbf{I} + \hat{\mathbf{G}}_c\hat{\mathbf{G}}_K)^{-1}\,\hat{\mathbf{G}}_c \tag{41}$$

By analogy with the definition of $\mathbf{G}^*$ in the Laplace domain, $\hat{\mathbf{Q}}^*$ is the matrix $\hat{\mathbf{Q}}$ with the delayed arguments (e.g. z^{-p_i}) eliminated. The overall transfer function for the discrete control system then follows as

$$\hat{\mathbf{y}}(z) = (\mathbf{I} + \hat{\mathbf{Q}}\,\hat{\mathbf{G}}_c^*\,\hat{\mathbf{H}})^{-1}\,[\hat{\mathbf{Q}}\,\hat{\mathbf{G}}_c^*\,\hat{\mathbf{y}}_d + \hat{\mathbf{Q}}_d\,\hat{\mathbf{d}}] \tag{42}$$

Equation (42) has a form analogous to (21), with $\mathbf{G}(s)$ and $\mathbf{G}_d(s)$ replaced by their respective discrete analogues $\hat{\mathbf{Q}}(z)$ and $\hat{\mathbf{Q}}_d(z)$. Thus, following precisely the same argument as in (21)-(26), we arrive at

$$\hat{\mathbf{y}}(z) = \hat{\mathbf{Q}}[\mathbf{I} + \hat{\mathbf{G}}_c\hat{\mathbf{H}}^*\hat{\mathbf{Q}}^*]^{-1}\,\hat{\mathbf{G}}_c\hat{\mathbf{y}}_d + \hat{\mathbf{Q}}[\mathbf{I} + \hat{\mathbf{G}}_c\hat{\mathbf{H}}^*\hat{\mathbf{Q}}^*]\hat{\mathbf{R}}\hat{\mathbf{Q}}^{-1}\,\hat{\mathbf{Q}}_d\hat{\mathbf{d}} \tag{43}$$

where

$$\hat{\mathbf{R}} = \mathbf{I} + \hat{\mathbf{G}}_c(\hat{\mathbf{H}}^*\hat{\mathbf{Q}}^* - \hat{\mathbf{H}}\hat{\mathbf{Q}}) \tag{44}$$

We note that the characteristic equation in (43) is similarly identified as

$$|\mathbf{I} + \hat{\mathbf{G}}_c\hat{\mathbf{H}}^*\hat{\mathbf{Q}}^*| = 0$$

and that it does not contain time delays.

SOME EXAMPLE PROBLEMS

To illustrate the performance of the multidelay compensator we shall consider some problems representative of engineering practice.

Example 1. Chemical Reactor Train With Recycle

To illustrate the case of state variable delays combined with output delays, we consider the two stage chemical reactor with recycle, shown in Figure 4. The irreversible reaction $A \to B$ with negligible heat effect is carried out in the two stage reactor system. Reactor temperature is maintained constant so that only the composition of product streams from the two reactors c_1, c_2 need be controlled. There is, however, substantial analysis delay. The manipulated variables are the feed compositions to the two reactors c_{1f}, c_{2f} and the process disturbance is an extra feed stream, F_d, whose composition c_d varies because it comes from another processing unit. The flow rates to the reactor system are fixed and only the compositions vary. The state delay arises due to the transportation lag in the recycle stream.

A material balance on the reactor train yields

$$V_1 \frac{dc_1}{dt} = F_1 c_{1f} + R c_2(t-\alpha) + F_d C_d - (F_1 + R + F_d) c_1 - V_1 k_1 c_1 \quad (45)$$

$$V_2 \frac{dc_2}{dt} = (F_1 + R + F_d - F_{p1}) c_1 + F_2 c_{2f} - (F_{p2} + R) c_2 - V_2 k_2 c_2 \quad (46)$$

where the second product stream, F_{p2} is given by

$$F_{p2} = F_1 + F_d - F_{p1} + F_2.$$

Defining the variables

$$\theta_1 = \frac{V_1}{F_1 + R + F_d}, \quad \theta_2 = \frac{V_2}{F_{p2} + R},$$

$$\lambda_R = \frac{R}{F_1 + R + F_d}, \quad \mu = \frac{F_{p2} - F_2 + R}{F_{p2} + R}$$

$$\lambda_d = \frac{F_d}{F_1 + R + F_d}, \quad Da_1 = k_1\theta_1, \quad Da_2 = k_2\theta_2$$

$$u_1 = c_{1f} - c_{1f_s}, \quad u_2 = c_{2f} - c_{2f_s}$$

$$x_1 = c_1 - c_{1s}, \quad x_2 = c_2 - c_{2s}, \quad d = c_d - c_{d_s}$$

(where c_{1f_s}, c_{2f_s}, c_{1s}, c_{2s}, c_{d_s} denote steady-state values) allows one to use vector-matrix notation so that (45, 46) become

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}_0\mathbf{x}(t) + \mathbf{A}_1\mathbf{x}(t-\alpha) + \mathbf{B}\mathbf{u}(t) + \mathbf{D}d \quad (47)$$

$$\left.\begin{array}{l} \mathbf{y}(t) = \mathbf{x}(t) \\ \mathbf{y}_m(t) = \mathbf{H}\mathbf{x}(t) \end{array}\right\}$$

where $\mathbf{y}_m(t)$ is the measured output and

$$\mathbf{A}_0 = \begin{bmatrix} -\frac{(1+Da_1)}{\theta_1} & 0 \\ \frac{\mu}{\theta_2} & -\frac{(1+Da_2)}{\theta_2} \end{bmatrix} \quad \mathbf{A}_1 = \begin{bmatrix} 0 & \frac{\lambda_R}{\theta_1} \\ 0 & 0 \end{bmatrix}$$

$$\mathbf{H} = \begin{bmatrix} e^{-\tau_1 s} & 0 \\ 0 & e^{-\tau_2 s} \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} \frac{(1-\lambda_R-\lambda_d)}{\theta_1} & 0 \\ 0 & \frac{(1-\mu)}{\theta_2} \end{bmatrix} \quad \mathbf{D} = \begin{bmatrix} \frac{\lambda_d}{\theta_1} \\ 0 \end{bmatrix}$$

Here τ_1, τ_2 are the delays in analyzing c_1 and c_2 respectively. Taking Laplace transforms, one obtains a transfer function model of the form

$$\overline{\mathbf{y}}(s) = \mathbf{G}(s)\overline{\mathbf{u}}(s) + \mathbf{G}_d(s)\overline{\mathbf{d}}(s) \quad (48)$$

or, since what we observe is $\overline{\mathbf{y}}_m$

$$\overline{\mathbf{y}}_m(s) = \mathbf{H}(s)\mathbf{G}(s)\overline{\mathbf{u}}(s) + \mathbf{H}(s)\mathbf{G}_d(s)\overline{\mathbf{d}}(s) \quad (49)$$

where

$$\mathbf{G}(s) = \frac{\begin{bmatrix} \frac{(1-\lambda_R-\lambda_d)}{\theta_1}\left[s + \frac{1+Da_2}{\theta_2}\right] & \frac{\lambda_R(1-\mu)}{\theta_1\theta_2} e^{-\alpha s} \\ \frac{(1-\lambda_R-\lambda_d)}{\theta_1\theta_2}\mu & \frac{(1-\mu)}{\theta_2}\left[s + \frac{1+Da_1}{\theta_1}\right] \end{bmatrix}}{s^2 + \left[\frac{(1+Da_1)}{\theta_1} + \frac{(1+Da_2)}{\theta_2}\right]s + \frac{(1+Da_1)(1+Da_2)}{\theta_1\theta_2} - \frac{\lambda_R\mu e^{-\alpha s}}{\theta_1\theta_2}} \quad (50)$$

$$\mathbf{G}_d(s) = \frac{\begin{bmatrix} \frac{\lambda_d}{\theta_1}\left(s + \frac{1+Da_2}{\theta_2}\right) \\ \frac{\lambda_d}{\theta_1}\frac{\mu}{\theta_2} \end{bmatrix}}{s^2 + \left[\frac{(1+Da_1)}{\theta_1} + \frac{(1+Da_2)}{\theta_2}\right]s + \frac{(1+Da_1)(1+Da_2)}{\theta_1\theta_2} - \frac{\lambda_R\mu e^{-\alpha s}}{\theta_1\theta_2}} \quad (51)$$

and if we let

$$\mathbf{G}_0(s) = \mathbf{H}(s)\mathbf{G}(s), \quad \mathbf{G}_{d0}(s) = \mathbf{H}(s)\mathbf{G}_d(s) \quad (52)$$

then the working transfer function matrices are

$$\mathbf{G}_0(s) = \frac{\begin{bmatrix} \frac{(1-\lambda_R-\lambda_d)}{\theta_1}\left[s + \frac{1+Da_2}{\theta_2}\right]e^{-\tau_1 s} & \frac{\lambda_R(1-\mu)}{\theta_1\theta_2} e^{-(\tau_1+\alpha)s} \\ \frac{(1-\lambda_R-\lambda_d)\mu e^{-\tau_2 s}}{\theta_1\theta_2} & \frac{(1-\mu)}{\theta_2}\left[s + \frac{(1+Da_1)}{\theta_1}\right]e^{-\tau_2 s} \end{bmatrix}}{s^2 + \left[\frac{(1+Da_1)}{\theta_1} + \frac{(1+Da_2)}{\theta_2}\right]s + \frac{(1+Da_1)(1+Da_2)}{\theta_1\theta_2} - \frac{\lambda_R\mu e^{-\alpha s}}{\theta_1\theta_2}} \quad (53)$$

Now using the multidelay compensator shown in Figure

3 and recalling the definitions of $\mathbf{H}^*$, $\mathbf{G}^*$, $\mathbf{H}^*$ now becomes $\mathbf{I}$ and

$$\mathbf{G}_{d0}(s) = \frac{\begin{bmatrix} \dfrac{\lambda_d}{\theta_1}\left(s + \dfrac{1 + Da_2}{\theta_2}\right) e^{-\tau_1 s} \\ \dfrac{\lambda_d}{\theta_1}\,\dfrac{\mu}{\theta_2}\, e^{-\tau_2 s} \end{bmatrix}}{s^2 + \left[\dfrac{(1 + Da_1)}{\theta_1} + \dfrac{(1 + Da_2)}{\theta_2}\right] s + \dfrac{(1 + Da_1)(1 + Da_2)}{\theta_1\theta_2} - \dfrac{\lambda_R \mu e^{-\alpha s}}{\theta_1\theta_2}} \tag{54}$$

$$\mathbf{G}^* = \mathbf{G}_0{}^* = \frac{\begin{bmatrix} \dfrac{(1 - \lambda_R - \lambda_d)}{\theta_1}\left[s + \dfrac{1 + Da_2}{\theta_2}\right] & \dfrac{\lambda_R(1 - \mu)}{\theta_1\theta_2} \\ \dfrac{(1 - \lambda_R - \lambda_d)\mu}{\theta_1\theta_2} & \dfrac{(1 - \mu)}{\theta_2}\left[s + \dfrac{(1 + Da_1)}{\theta_1}\right] \end{bmatrix}}{s^2 + \left[\dfrac{(1 + Da_1)}{\theta_1} + \dfrac{(1 + Da_2)}{\theta_2}\right] s + \dfrac{(1 + Da_1)(1 + Da_2)}{\theta_1\theta_2} - \dfrac{\lambda_R \mu}{\theta_1\theta_2}} \tag{55}$$

and by choosing two single loop proportional controllers for $\mathbf{G}_c$ i.e.,

$$\mathbf{G}_c = \begin{bmatrix} k_{11c} & 0 \\ 0 & k_{22c} \end{bmatrix} \tag{56}$$

we may compare the control system performance both with and without the compensator. To illustrate, let us choose

$$\theta_1 = 1, \quad \theta_2 = 1, \quad Da_1 = 1, \quad Da_2 = 1,$$

$$\lambda_R = 0.5, \quad \lambda_d = 0.1, \quad \mu = 0.5$$

with time delays $\alpha = 1$, $\tau_1 = 3$, $\tau_2 = 2$. In this case

$$\mathbf{G}_0(s) = \mathbf{H}(s)\mathbf{G}(s) = \begin{bmatrix} \dfrac{0.4(s + 2)e^{-3s}}{(s + 2)^2 - 0.25e^{-s}} & \dfrac{0.25e^{-4s}}{(s + 2)^2 - 0.25e^{-s}} \\ \dfrac{0.2e^{-2s}}{(s + 2)^2 - 0.25e^{-s}} & \dfrac{0.5(s + 2)e^{-2s}}{(s + 2)^2 - 0.25e^{-s}} \end{bmatrix} \tag{57}$$

$$\mathbf{G}_0{}^*(s) = \mathbf{H}^*(s)\mathbf{G}^*(s) = \begin{bmatrix} \dfrac{0.4(s + 2)}{(s + 2.5)(s + 1.5)} & \dfrac{0.25}{(s + 2.5)(s + 1.5)} \\ \dfrac{0.2}{(s + 2.5)(s + 1.5)} & \dfrac{0.5(s + 2)}{(s + 2.5)(s + 1.5)} \end{bmatrix} \tag{58}$$

$$\mathbf{G}_d(s) = \begin{bmatrix} \dfrac{0.1(s + 2)e^{-3s}}{(s + 2)^2 - 0.25e^{-s}} \\ \dfrac{0.05e^{-2s}}{(s + 2)^2 - 0.25e^{-s}} \end{bmatrix} \tag{59}$$

so that the closed loop system *without the compensator* is

$$\mathbf{y} = [\mathbf{I} + \mathbf{G}\mathbf{G}_c\mathbf{H}]^{-1}\,\mathbf{G}\mathbf{G}_c\mathbf{y}_d + [\mathbf{I} + \mathbf{G}\mathbf{G}_c\mathbf{H}]^{-1}\,\mathbf{G}_d\mathbf{d} \tag{60}$$

or equivalently, since $\mathbf{y}_m$ is observed

$$\mathbf{y}_m = [\mathbf{I} + \mathbf{H}\mathbf{G}\mathbf{G}_c]^{-1}\,\mathbf{H}\mathbf{G}\mathbf{G}_c\mathbf{y}_d + [\mathbf{I} + \mathbf{H}\mathbf{G}\mathbf{G}_c]^{-1}\,\mathbf{H}\mathbf{G}_d\mathbf{d}$$

or in terms of $\mathbf{G}_0$, $\mathbf{G}_{d0}$,

$$\mathbf{y}_m = [\mathbf{I} + \mathbf{G}_0\mathbf{G}_c]^{-1}\,\mathbf{G}_0\mathbf{G}_c\mathbf{y}_d + [\mathbf{I} + \mathbf{G}_0\mathbf{G}_c]^{-1}\,\mathbf{G}_{d0}\mathbf{d} \tag{61}$$

Now *with the compensator* the closed loop expressions are

$$\mathbf{y} = \mathbf{G}(\mathbf{I} + \mathbf{G}_c\mathbf{H}^*\mathbf{G}^*)^{-1}\,\mathbf{G}_c\mathbf{y}_d + \mathbf{G}(\mathbf{I} + \mathbf{G}_c\mathbf{H}^*\mathbf{G}^*)^{-1}\,\mathbf{R}\mathbf{G}^{-1}\,\mathbf{G}_d\mathbf{d}$$

where $\mathbf{R} = \mathbf{I} + \mathbf{G}_c(\mathbf{H}^*\mathbf{G}^* - \mathbf{H}\mathbf{G})$, or in terms of $\mathbf{G}_0$, $\mathbf{G}_{d0}$,

$$\mathbf{R} = \mathbf{I} + \mathbf{G}_c(\mathbf{G}_0{}^* - \mathbf{G}_0)$$

$$\mathbf{y}_m = \mathbf{G}_0(\mathbf{I} + \mathbf{G}_c\mathbf{G}_0{}^*)^{-1}\,\mathbf{G}_c\mathbf{y}_d + \mathbf{G}_0(\mathbf{I} + \mathbf{G}_c\mathbf{G}_0{}^*)\mathbf{R}\mathbf{G}^{-1}\,\mathbf{G}_d\mathbf{d} \tag{62}$$

A minimal realization for $\overline{\mathbf{y}}^*(s) = \mathbf{G}_0{}^*(s)\,\overline{\mathbf{u}}(s)$ is easily shown to be

$$\dot{x}_1{}^*(t) = -2.5x_1{}^*(t) + 0.2u_1 - 0.25u_2 \tag{63}$$

$$\dot{x}_2{}^*(t) = -1.5x_2{}^*(t) + 0.2u_1 + 0.25u_2 \tag{64}$$

with

$$y_1{}^*(t) = x_1{}^*(t) + x_2{}^*(t) \tag{65}$$

$$y_2{}^*(t) = x_2{}^*(t) - x_1{}^*(t) \tag{66}$$

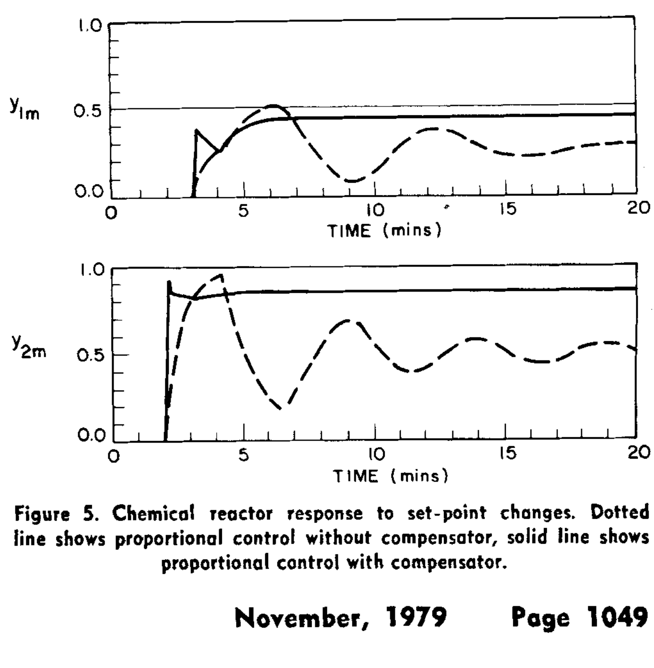

Figure 5. Chemical reactor response to set-point changes. Dotted line shows proportional control without compensator, solid line shows proportional control with compensator.

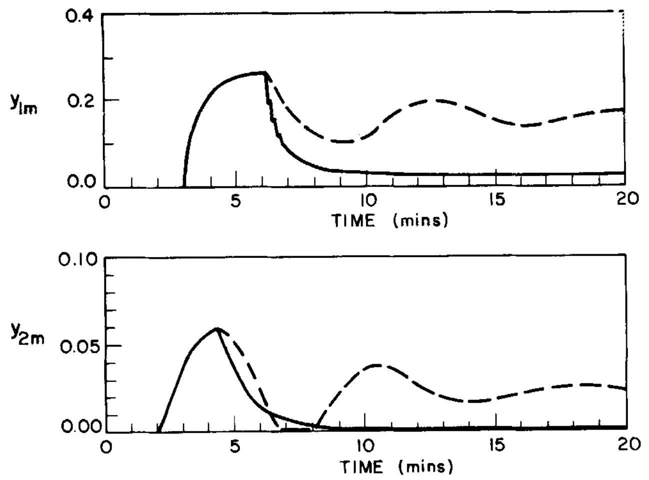

Figure 6. Chemical reactor response to a step change disturbance. Dotted line shows proportional control without compensator, solid line shows proportional control with compensator.

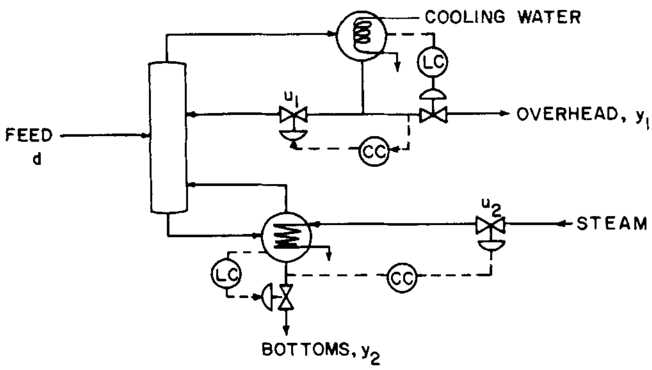

Figure 7. Schematic diagram of the methanol distillation column with conventional two point column control system (Wood & Berry 1973).

The control system performance for set point changes $y_{1d} = 0.5$, $y_{2d} = 1.0$ are shown in Figure 5. The dashed lines represent the performance without the compensator for controller gains $k_{11c} = 3.0$, $k_{22c} = 3.5$. In the neighborhood of $K_c = 5.0$, serious instabilities set in.

The application of the multidelay compensator permits the use of higher controller gains $k_{11c} = k_{22c} = 20.0$, and as seen in Figure 5 (solid lines), performance is greatly improved.

In Figure 6, we see the response to a step input in disturbance $d = 5.0$. The dashed lines show the response without the compensator. Again, controller gains $k_{11c} = 3.0$, $k_{22c} = 3.5$ are used, these being the largest before the onset of serious instabilities. The continuous lines show performance using the compensator with gains $k_{11c} = 45.0$, $k_{22c} = 20.0$. Observe, from the modelling Equations (47, 48) that the influence of the disturbance on x_1 is direct, while x_2 is affected only as a result of its coupling with x_1. Hence with $k_{22c} = 20.0$, x_2 has essentially attained steady state (with negligible offset), while higher values had to be used for k_{11c} to considerably reduce the offset in x_1. Serious instabilities set in for k_{11c} values in the neighborhood of 55.0.

This example serves to illustrate the improvements in control with the multidelay compensator for a problem having both state and measurement delays.

Example 2. Binary Distillation Column

To illustrate the effects of multiple delays in the control and output variables, let us consider the binary distillation column studied by Wood and Berry (1973), Shah and Fisher (1978), and Meyer et al. (1978, 1979). The column, shown in Figure 7, was used for methanol-water separation and was well modelled by the transfer function model

$$\overline{y}(s) = \mathbf{G}(s)\overline{u}(s) + \mathbf{G}_d(s)\overline{d}(s) \tag{67}$$

where, in terms of deviation variables, y_1 = overhead mole fraction methanol, y_2 = bottoms mole fraction methanol, u_1 = overhead reflux flow rate, u_2 = bottoms steam flow rate, d = column feed flow rate. After pulse testing the column, the transfer function matrices determined from the data were (Wood and Berry 1973)

$$\mathbf{G}(s) = \begin{bmatrix} \dfrac{12.8\,e^{-s}}{16.7s+1} & \dfrac{-18.9\,e^{-3s}}{21.0s+1} \\[2ex] \dfrac{6.6\,e^{-7s}}{10.9s+1} & \dfrac{-19.4\,e^{-3s}}{14.4s+1} \end{bmatrix} \tag{68}$$

$$\mathbf{G}_d(s) = \begin{bmatrix} \dfrac{3.8\,e^{-8.1s}}{14.9s+1} \\[2ex] \dfrac{4.9\,e^{-3.4s}}{13.2s+1} \end{bmatrix} \tag{69}$$

where the time constants and time delays are given in minutes. Here we take $\mathbf{H} = \mathbf{H}^* = \mathbf{I}$ because any measurement delays may be included in $\mathbf{G}$ for this problem. —We shall again choose to illustrate the performance of the system under conventional PI control (Figure 7), both with and without the compensator.

The computer simulation requires a state-space representation of the model. Following the procedure outlined by Ogunnaike (in press), a minimal realization for Equation (67), given transfer functions in (68) and (69) is readily obtained as

$$\left.\begin{aligned} \dot{x}_1(t) &= -0.06\,x_1(t) + 0.768\,u_1(t-1) \\ \dot{x}_2(t) &= -0.04762\,x_2(t) + 0.9\,u_2(t-3) \\ \dot{x}_3(t) &= -0.09174\,x_3(t) + 0.6055\,u_1(t-7) \\ \dot{x}_4(t) &= -0.06944\,x_4(t) + 1.3472\,u_2(t-3) \end{aligned}\right\} \tag{70}$$

along with

$$\left.\begin{aligned} \dot{x}_{L1}(t) &= -0.0671\,x_{L1}(t) + 0.255\,d(t-8.1) \\ \dot{x}_{L2}(t) &= -0.07576\,x_{L2}(t) + 0.3712\,d(t-3.4) \end{aligned}\right\} \tag{71}$$

and

$$\left.\begin{aligned} y_1(t) &= x_1(t) - x_2(t) + x_{L1} \\ y_2(t) &= x_3(t) - x_4(t) + x_{L2} \end{aligned}\right\} \tag{72}$$

where $x_i(t)$ is a member of the vector of state variables ($i = 1, \ldots 4$) and $x_{Li}(t)$ is a member of the load disturbance 'state' vector; $y_i(t)$ is the output.

The steady state values for the overhead and bottoms compositions are taken to be 96.25% and 0.5% methanol respectively (cf., Wood and Berry 1973) for this simulation. With the "tuned" conventional controller settings originally used experimentally by Wood and Berry (1973) and reported in their Table 1*,

Overhead		Bottoms	
K_p	K_I	K_p	K_I
0.20	0.045	−0.040	−0.015

* Private communications with R. K. Wood confirmed that the signs of the controller gains reported in the original publication should be corrected as shown here.

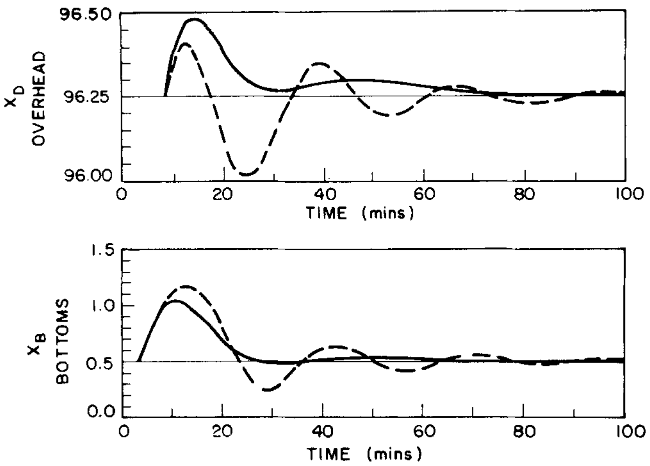

Figure 8. Comparison of column performance with and without the multidelay compensator (positive feed rate disturbance). Dotted line shows performance without compensator, solid line shows performance with compensator.

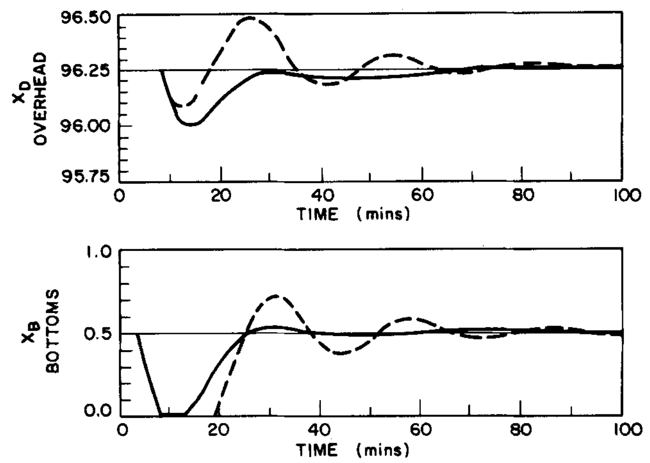

Figure 9. Comparison of column performance with and without the multidelay compensator (negative feed rate disturbance). Dotted line shows performance without compensator, solid line, with compensator.

the control system response is shown as dashed lines in Figures 8 and 9. Figure 8 is the response to the same positive disturbance 0.34 lb/min in feed flow rate as that used in the experimental study. Figure 9 is the response to the negative disturbance (−0.36 lb/min) in feed flow rate.

This simulated response is essentially identical to the experimental response reported by Wood and Berry (1973) for conventional control. Larger controller gains cannot be taken because the characteristic equation

$$|\mathbf{I} + \mathbf{G}\mathbf{G}_c| = 55045\,s^4 + 14698\,s^3 + 1219\,s^2 + 62s + 1 + (228.9s^2 + 31.9s + 1)\,[g_{c1}(12 + 172.8s)e^{-s} - g_{c2}(19.4 + 323.8s)e^{-3s} - 232.8g_{c1}g_{c2}\,e^{-4s}] + 124.7\,(240.5\,s^2 + 31.1s + 1)g_{c1}g_{c2}\,e^{-10s} = 0$$

contains time delays which cause stability problems. Here we have taken

$$\mathbf{G}_c = \begin{bmatrix} g_{c1} & 0 \\ 0 & g_{c2} \end{bmatrix} \tag{73}$$

with $g_{ci} = K_{pi} + K_{Ii}/s$ in keeping with the notation of Wood and Berry. When the multidelay compensator is applied to the control loop as in Figure 3 with

$$\mathbf{G}^* = \begin{bmatrix} \dfrac{12.8}{16.7s + 1} & \dfrac{-18.9}{21.0s + 1} \\[2ex] \dfrac{6.6}{10.9s + 1} & \dfrac{-19.4}{14.4s + 1} \end{bmatrix}$$

and $\mathbf{G}_c$ as in (73), the characteristic equation becomes

$$|\mathbf{I} + \mathbf{G}_c\mathbf{G}^*| = 55045\,s^4 + (14698 + 39553\,g_{c1} - 74117\,g_{c2})s^3 + (1219 + 8259\,g_{c1} - 14769\,g_{c2} - 23290\,g_{c1}g_{c2})s^2 + (62 + 555\,g_{c1} - 942\,g_{c2} - 3546\,g_{c1}g_{c2})s + (1 - 108\,g_{c1}g_{c2}) = 0$$

and contains no time delays. The improved response obtained with the compensator is shown as continuous lines in Figures 8 and 9 for precisely the same controller settings used with the PI controller alone. Apart from noting that there are no serious oscillations, observe that the bottoms composition benefits more from the use of the multidelay compensator. That this should indeed be so is readily seen by merely inspecting the transfer function matrix $\mathbf{G}(s)$ and noting that the time delays associated with the bottoms are substantially larger than those associated with the overhead.

One interesting feature of this distillation column is the appreciable amount of interactions existing between the system variables. This is due to the presence of off-diagonal elements in $\mathbf{G}(s)$ with the result that y_1 is affected by u_2 and y_2 by u_1. The system performance is most affected by this coupling when set-point changes are made. For example, when a set-point change from 96.25 to 97.0 is made in the overhead composition, the multidelay compensated system responds as shown in dashed lines in Figure 10. (The controller gains used here are

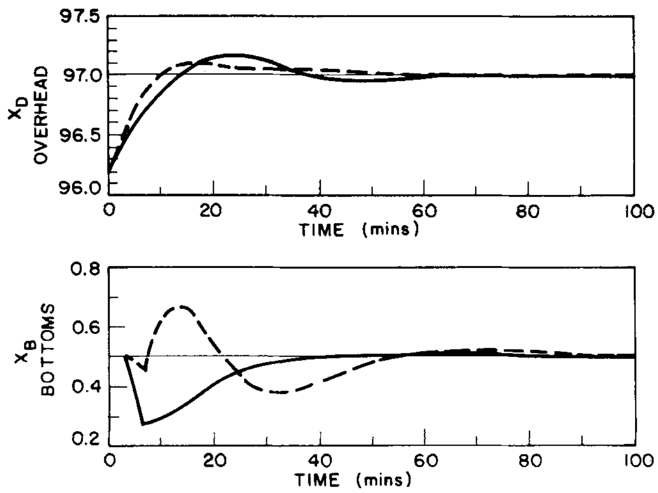

Figure 10. Column response to a positive set-point change in overhead composition using the multidelay compensator. Dotted line, without steady-state decoupling; solid line, with steady-state decoupling.

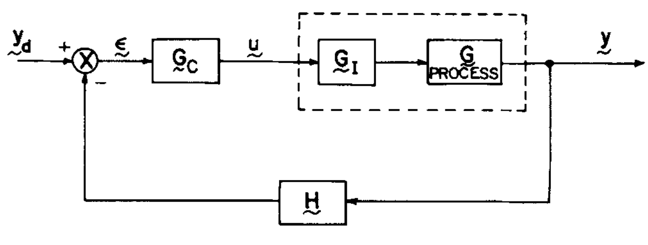

Figure 11. The decoupling compensator.

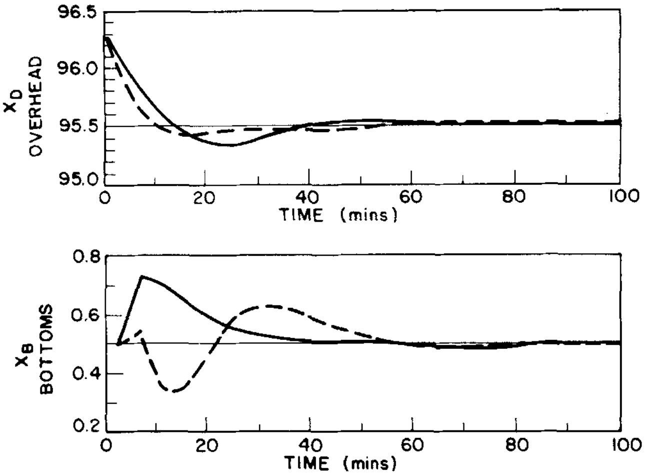

Figure 12. Column response to a negative set-point change in overhead composition using the multidelay compensator. Dotted line, without steady-state decoupling; solid line, with steady-state decoupling.

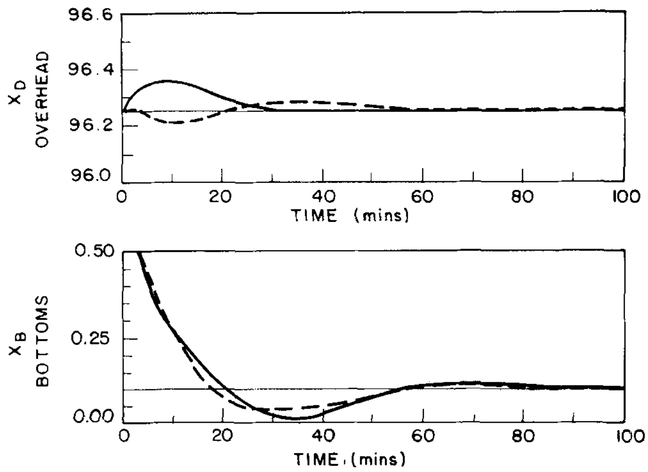

Figure 13. Column response to a positive set-point change in bottoms composition using the multidelay compensator. Dotted line, without steady-state decoupling; solid line, with steady-state decoupling.

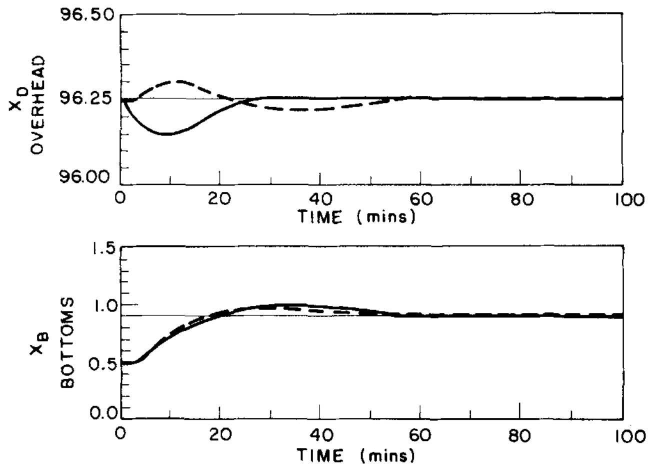

Figure 14. Column response to a negative set-point change in bottoms composition using the multidelay compensator. Dotted line, without steady-state decoupling; solid line, with steady-state decoupling.

the same as those reported earlier.) The interesting point to note is the resulting effect on the bottoms composition. Were there no coupling between the overhead and bottoms compositions, such a set-point change would not have perturbed the bottoms composition. The simplest first step in dealing with such interaction is to attempt to eliminate the steady state effects through the use of steady-state decoupling, along with the multidelay compensator.

The philosophy of the steady-state decoupler (cf., Ray, in press) is to introduce a compensator block $\mathbf{G}_I$ (see Figure 11) such that $\mathbf{G}_I\mathbf{G} = \mathbf{I}$ at steady state and hence the interactions between the system variables are eliminated. $\mathbf{G}_I$ in this case is the steady state inverse of $\mathbf{G}$ and is usually combined with $\mathbf{G}_c$ resulting in a controller given by

$$\mathbf{G}_c' = \mathbf{G}_I\mathbf{G}_c$$

For this example problem

$$\mathbf{G}_I = \begin{bmatrix} 0.157 & -0.153 \\ 0.053 & -0.1036 \end{bmatrix}$$

When the following controller settings are used

Overhead		Bottoms	
K_p	K_I	K_p	K_I
0.5	0.5	0.6	0.1

the response using both the multidelay compensator and the steady-state decoupler is shown as continuous lines in Figure 10. There is now no need to use negative controller gains for the bottoms since the effective result of including $\mathbf{G}_I$ is that $\mathbf{G}_I\mathbf{G}$ combine to yield $\mathbf{I}$ at steady state. Negative gains were originally used because of the negative steady state gains in column 2 of $\mathbf{G}$.

Note the faster return of the bottoms composition to steady state. For a similar overhead set-point change in the opposite direction (from 96.25 to 95.5) the system response (using the same controller settings noted above) is shown in Figure 12 where the continuous lines show the response when the multidelay compensator is used along with the steady-state decoupler.

When set-point changes are made in the bottoms composition, the responses are shown in Figures 13 and 14, where the set-points have been changed from 0.5 to 0.9, and from 0.5 to 0.1, respectively. Here the controller settings are the same as before. In this case, the overhead composition is restored to its desired value somewhat faster with steady state decoupling. It was found that the use of steady state decoupling had very little effect on the control system response to disturbances in the feed rate. This might be expected since the steady state effect of feed rate disturbances on both overhead and bottoms composition is approximately the same (cf., Equation (69)).

As noted above, the simulated responses using the multidelay compensator are much better than those reported by Wood and Berry (1973) using conventional, ratio, and non-interacting control schemes on the same column. This comparison may be somewhat unfair because Wood and Berry report experimental controller responses and our results are based on simulations using their model. Nevertheless, this example problem aptly illustrates the potential value of the multidelay compensator in distillation column control problems.

NOTATION

$\mathbf{A}, \mathbf{A}_0, \mathbf{A}_1, \mathbf{A}_i$ = state-vector matrix coefficients (in state-space model)

$\mathbf{B}, \mathbf{B}_j$ = control vector matrix coefficients (in state-space model)
$\mathbf{C}, \mathbf{C}_i$ = output matrix coefficients (in state-space model)
c_1, c_2 = reacting fluid concentrations
$\mathbf{D}, \mathbf{D}_k$ = disturbance matrix coefficient
Da_1, Da_2 = Damköhler numbers
d = disturbance vector
$\mathbf{E}_j$ = matrix coefficient (in state-space model)
$F_1, F_2, F_{p1}, F_{p2}, F_d$ = flow rates of reactor streams
f_{ijq} = transfer function pole
$\mathbf{G}, \mathbf{G}_0, \mathbf{G}^*$ = transfer function matrices
$\mathbf{G}_c, \mathbf{G}_c{}^*$ = controller transfer functions
$\mathbf{G}_d, \mathbf{G}_{d0}$ = load disturbance transfer function
$\mathbf{G}_I$ = interaction compensator
$\mathbf{G}_K$ = multidelay compensator transfer function
g_{ij} = elements of arbitrary transfer function
$\mathbf{H}, \mathbf{H}^*$ = measurement transfer function
h_{ijp} = transfer function zero
$\mathbf{I}$ = identity matrix
k_{ij} = steady state gain
K_p = proportional (controller) gain
K_I = integral "gain" ($K_I = K_p/\tau_I$)
$\mathbf{L}$ = matrix coefficient
$\mathbf{P}$ = matrix coefficient
$\hat{\mathbf{Q}}, \hat{\mathbf{Q}}_d$ = z-transform transfer function
$\mathbf{R}$ = combination of transfer function matrices
s = Laplace operator
T_s = sampling time
$\hat{\mathbf{u}}$ = z-transform of $\mathbf{u}(s)$
$\mathbf{u}$ = control vector
$\hat{\mathbf{X}}$ = z-transform of $\mathbf{x}(s)$
$\mathbf{x}, \mathbf{x}^*$ = state-vector
x_d, x_B = overheads and bottoms compositions
$\hat{\mathbf{y}}$ = z-transform of $y(s)$
$\mathbf{y}$ = output vector
$\mathbf{y}^*$ = output from multidelay compensator
y_m = measured output
z = z-transform operator

Greek Letters

α = time delay in reactor state variable
α_{ij} = time delay in transfer function elements
β_j = time delay in control path
γ_i = time delay in output
$\mathbf{\Gamma}$ = closed loop transfer function
$\mathbf{\Delta}_j$ = matrix coefficient in discrete time state-space model
δ_k = time delay in input disturbance
ε = error signal
ϵ_j = time delay in control associated with output
$\mathbf{\Phi}_i$ = matrix coefficient in discrete time state-space model
λ_R, λ_d = dimensionless flow ratios
μ = dimensionless flow ratio
$\mathbf{\Theta}_k$ = matrix coefficient in discrete time state-space model
θ = dummy time argument
θ_1, θ_2 = reactor space times
ρ_i = time delay in state variable
τ = time constant
τ_1, τ_2 = time delays in measurement

Subscripts

f = feed
s = steady state

Superscripts

$*$ = multiple delay compensator quantities; matrices without their delays
$\wedge$ = z transformed variables
— = Laplace transformed variables

APPENDIX A. CHARACTERISTIC EQUATION FOR THE MULTIDELAY COMPENSATED SYSTEM

It is well known that for single input, single output systems, stability is determined from the nature of the roots of the characteristic equation

$$1 + GH = 0 \tag{A-1}$$

where G is the transfer function in the forward path. However, in forming the transfer function G, pole-zero cancellations can occur and hence (A-1) does not give information about the location of the cancelled poles which may be stable or unstable. A similar cancellation can occur in multivariable systems (see Rosenbrock 1970, 1974 for examples) and stability may then be inferred only with additional assumptions.

Consider the system (with no time delays) in Figure 1. Under conventional control,

$$\bar{\mathbf{y}}(s) = (\mathbf{I} + \mathbf{G}\mathbf{G}_c\mathbf{H})^{-1}[\mathbf{G}\mathbf{G}_c\bar{\mathbf{y}}_d(s) + \mathbf{G}_d\bar{\mathbf{d}}(s)] \tag{A-2}$$

if we define

$$\mathbf{\Gamma} = (\mathbf{I} + \mathbf{G}\mathbf{G}_c\mathbf{H})^{-1}\mathbf{G}\mathbf{G}_c \text{ (closed loop transfer function)} \tag{A-3}$$

Stability requires that the roots of the equation

$$|(\mathbf{I} + \mathbf{G}\mathbf{G}_c\mathbf{H})| = 0 \tag{A-4}$$

all lie in the left half plane (see Rosenbrock 1970, 1974, 1975 for various other alternative statements).
Now, it is clear from the Equation (A-3) that

$$|(\mathbf{I} + \mathbf{G}\mathbf{G}_c\mathbf{H})| = |\mathbf{G}\mathbf{G}_c\mathbf{\Gamma}^{-1}| = \frac{|\mathbf{\Gamma}^{-1}|}{|(\mathbf{G}\mathbf{G}_c)^{-1}|} \tag{A-5}$$

Hence (A-4) implies that

$$\frac{|\mathbf{\Gamma}^{-1}|}{|(\mathbf{G}\mathbf{G}_c)^{-1}|} = 0 \tag{A-6}$$

which in general would lead us to the conclusion that

$$|\mathbf{\Gamma}^{-1}| = 0 \tag{A-7}$$

However, we note that both in the formation of $\mathbf{\Gamma}$ and $(\mathbf{G}\mathbf{G}_c)$, cancellations can occur. Extra conditions are therefore needed to ensure that the cancelled poles are not unstable. Following Rosenbrock (1970) a sufficient condition to ensure this is that the open loop system be asymptotically stable. (Although this condition is usually satisfied in process control problems, more detailed analysis is necessary if open loop stability does not exist.)

If this additional condition holds, closed loop stability can be inferred from the nature of the roots of

$$|\mathbf{\Gamma}^{-1}| = 0$$

For the multidelay compensated system (Equation 26)

$$\mathbf{\Gamma} = \mathbf{G}(\mathbf{I} + \mathbf{G}_c\mathbf{H}^*\mathbf{G}^*)^{-1}\mathbf{G}_c \tag{A-8}$$

whence

$$|\mathbf{\Gamma}^{-1}| = 0$$

requires that

$$|\mathbf{G}_c{}^{-1}|\,|\mathbf{I} + \mathbf{G}_c\mathbf{H}^*\mathbf{G}^*|\,|\mathbf{G}^{-1}| = 0 \tag{A-9}$$

This leads to the characteristic equation

$$|\mathbf{I} + \mathbf{G}_c\mathbf{H}^*\mathbf{G}^*| = 0 \tag{A-10}$$

which is sufficient for stability, if there is no pole-zero cancellation or if the open loop system is stable.

APPENDIX B. THE SMITH PREDICTOR VERSUS THE ANALYTICAL PREDICTOR

It is clear that in the case of a single variable with a single delay, the multidelay compensator presented here reduces to the Smith Predictor. Let us now show that the Smith Predictor and the analytical predictor of Moore et al. (1970) are equivalent for a first order plus time delay system. Consider the system given by $h(s) = 1$ and

$$y(s) = g(s)e^{-\alpha s}\, u(s) \tag{B-1}$$

By applying the Smith Predictor (Figure 2), the minor loop introduces a signal

$$-\, g(s)\, [1 - e^{-\alpha s}]\, u(s)$$

which combines with that from the system, $-\, g(s)e^{-\alpha s}u(s)$ resulting in some $-\, y^*(s)$ given by

$$y^*(s) = g(s)\, u(s) \tag{B-2}$$

being sent to the controller. It is clear from (B-2) and (B-1) that

$$y^*(s) = e^{\alpha s}y(s)$$

or in the time domain.

$$y^*(t) = y(t + \alpha) \tag{B-3}$$

The so-called analytical predictor of Moore et al. (1970) was designed for first order processes having models of the form

$$\tau \frac{dy(t)}{dt} + y(t) = K_1 u(t - \alpha) \tag{B-4}$$

According to Moore et al., the discrete-time analytical predictor algorithm must calculate a control action at time t based on an output y^*, predicted for time $t + \alpha$ in the future where α is the actual dead time. Thus

$$y^*(t) = y(t + \alpha) \tag{B-5}$$

However, the analytical predictor has been used in the discrete form with a zero order hold element requiring an additional time prediction of $0.5T_s$ where T_s is the sampling interval. Thus as formulated by Moore et al. and in practice (Meyer et al. 1979) the analytical predictor takes the form

$$y^*(t_k) = y(t_k + \alpha + 0.5T_s) \tag{B-6}$$

where $t_k = NT_s$ is the current time.

Comparing Equations (B-3) and (B-5) demonstrates that in the continuous time domain, the Smith and Moore predictors are identical. Thus if both are implemented in the discrete time domain, they should also be identical (Equation B-6).

APPENDIX C. COMPARISON OF THE MULTIDELAY COMPENSATED CONTROLLER WITH THE OPTIMAL FEEDBACK CONTROLLER

Consider the usual transfer function model having input delays

$$\bar{\mathbf{y}}(s) = \begin{bmatrix} g_{11}(s)e^{-\gamma_{11}s} & g_{12}(s)e^{-\gamma_{12}s} & \cdots\cdots \\ g_{21}(s)e^{-\gamma_{21}s} & g_{22}(s)e^{-\gamma_{22}s} & \cdots\cdots \\ \vdots & \vdots & \\ \vdots & \vdots & \end{bmatrix} \bar{\mathbf{u}}(s)$$

If we define

$$\bar{z}_{ij}(s) = g_{ij}(s)e^{-\gamma_{ij}s}\,\bar{u}_j(s) \tag{C-1}$$

then

$$\bar{y}_i(s) = \sum_j g_{ij}e^{-\gamma_{ij}s}\,\bar{u}_j(s)$$

or in vector-matrix notation

$$\bar{\mathbf{y}}(s) = \mathbf{L}\bar{\mathbf{z}}(s) \tag{C-2}$$

where $\bar{\mathbf{z}}(s)$, and $\mathbf{L}$ are given by

$$\bar{\mathbf{z}}(s) = \begin{bmatrix} z_{11}(s) \\ z_{12}(s) \\ \vdots \\ z_{1j}(s) \\ \hline z_{21}(s) \\ z_{22}(s) \\ \vdots \\ \hline \vdots \\ \hline \vdots \\ z_{ij}(s) \end{bmatrix} ; \quad \mathbf{L} = \left.\begin{bmatrix} \overbrace{11 \;\cdots\cdots\; 1}^{j \text{ columns}} & 00 \;\cdots\; 0 & \cdots\cdots & 00 \;\cdots\; 00 \\ 00 \;\cdots\cdots\; 0 & 11 \;\cdots\; 1 & \cdots\cdots & 00 \;\cdots\; 00 \\ \vdots & \vdots & \vdots & \vdots \\ 00 \;\cdots\cdots\; 0 & 00 \;\cdots\; 0 & \cdots\cdots & 11 \;\cdots\; 11 \end{bmatrix}\right\} i \text{ rows}$$

The philosophy of the multidelay compensator is to cause control action to be taken based on

$$\bar{\mathbf{y}}^*(s) = \begin{bmatrix} g_{11}(s) & g_{12}(s) & \cdots\cdots \\ g_{21}(s) & g_{22}(s) & \cdots\cdots \\ \vdots & \vdots & \vdots \end{bmatrix} \bar{\mathbf{u}}(s) \tag{C-3}$$

from the definition of $\bar{z}_{ij}(s)$ in (C-1) it is then obvious that

$$\bar{y}_i^*(s) = \sum_j e^{\gamma_{ij}s}\,\bar{z}_{ij}(s) \tag{C-4}$$

or

$$\bar{\mathbf{y}}^*(s) = \mathbf{L}\bar{\mathbf{z}}^*(s) \tag{C-5}$$

where

$$
\bar{\mathbf{z}}^*(s) = \begin{bmatrix} e^{\gamma_{11}s}\bar{z}_{11}(s) \\ e^{\gamma_{12}s}\bar{z}_{12}(s) \\ \vdots \\ e^{\gamma_{1j}s}\bar{z}_{1j}(s) \\ \hdashline e^{\gamma_{21}s}\bar{z}_{21}(s) \\ \vdots \\ \hdashline \vdots \\ e^{\gamma_{ij}s}\bar{z}_{ij}(s) \end{bmatrix} \quad \text{(C-6)}
$$

We now find it useful, for clarity of presentation, to rank the delays $\alpha_k = \gamma_{ij}$ in ascending order i.e.

$$
\alpha_1 < \alpha_2 < \ldots . < \alpha_k < \ldots . < \alpha_n \quad \text{(C-7)}
$$

where

$$
n = i \times j
$$

Hence any particular delay γ_{lm} from the l^{th} row and m^{th} column of the original transfer function is now represented as α_k, ranking k^{th} in magnitude in accordance with (C-7). α_n is the largest delay time, α_1 is the smallest.

If the $\bar{z}_{lm}(s)$ associated with the delay term γ_{lm} (now cast as α_k) is represented as $\bar{x}_k(s)$ then (C-1), (C-5) and (C-6) become, respectively

$$
\bar{x}_k(s) = g_k(s)\, e^{-\alpha_k s}\,\bar{u}_m(s) \quad \text{(C-8)}
$$

$$
\bar{\mathbf{y}}^*(s) = \mathbf{C}'\bar{\mathbf{x}}^*(s) \quad \text{(C-9)}
$$

with

$$
\bar{\mathbf{x}}^*(s) = \begin{bmatrix} e^{\alpha_1 s}\bar{x}_1(s) \\ e^{\alpha_2 s}\bar{x}_2(s) \\ \vdots \\ e^{\alpha_n s}\bar{x}_n(s) \end{bmatrix} \quad \text{(C-10)}
$$

The matrix $\mathbf{C}'$ results from an appropriate rearrangement in $\mathbf{L}$ consequential to the ranking of the $\bar{\mathbf{z}}^*$ column to yield $\bar{\mathbf{x}}^*$.

We aim to show that the application of the multidelay compensator results in a feedback law which is essentially of the same structure as that of the optimal feedback controller for linear systems with multiple delays (Soliman and Ray 1972). The most general form $g_k(s)$ takes is

$$
g_k(s) = \frac{k_k \prod_{p=1}^{m} (h_{kp}s + 1)}{\prod_{q=1}^{r} (f_{kq}s + 1)} \quad \text{(C-11)}
$$

where the fact that, for a dynamical system, $g_k(s)$ should be a proper rational function requires that $m < r$.

We note that a partial fraction expansion of (C-11) yields the form*

$$
g_k(s) = k_k \sum_{i=1}^{r} \frac{\omega_i}{(f_i s + 1)} \quad \text{(C-12)}
$$

a summation of first order functions. As such, a $g_k(s)$ with the simple first order form

$$
g_k(s) = \frac{b_k}{s - a_k} \quad \text{(C-13)}
$$

preserves all the features contained in (C-11). For simplicity, then we will use (C-13) in (C-8) for $g_k(s)$, recognizing that (C-11) would only result in an expanded state vector, all of which are first order.

Transforming (C-8) to the time domain, then, on using (C-13) we obtain

$$
\dot{x}_k(t) = a_k x_k(t) + b_k u_m(t - \alpha_k) \quad \text{(C-14)}
$$

If there are n of the x_k variables, r inputs u, (C-14) can be rewritten in vector-matrix form as

$$
\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \sum_{i=1}^{n} \mathbf{B}_i \mathbf{u}(t - \alpha_i) \quad \text{(C-15)}
$$

where $\mathbf{A}$ is a square $n \times n$ matrix with diagonal elements $a_1, a_2, \ldots . a_n$.

$\mathbf{B}_i$ is an $n \times r$ matrix containing a single non-zero element b_i in the appropriate position (dictated by (C-14)), all other elements being zero. It is essential to keep this structure of the $\mathbf{B}_i$ matrix in mind.

Transforming (C-10) back to the time domain yields

$$
\mathbf{x}^*(t) = \begin{bmatrix} x_1(t + \alpha_1) \\ x_2(t + \alpha_2) \\ \vdots \\ x_k(t + \alpha_k) \\ \vdots \\ x_n(t + \alpha_n) \end{bmatrix} \quad \text{(C-16)}
$$

As previously noted, it is apparent that $\mathbf{x}^*(t)$ does not represent any actual value of the vector $\mathbf{x}(t)$ but a "collection" of individual predictions of each member of $\mathbf{x}(t)$ over various delay times. However, we note that the time evolution of each x_k is governed by (C-14), while (C-15) governs that for the vector $\mathbf{x}(t)$. The solution to (C-15) is readily shown to be

$$
\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0 + \int_0^t e^{\mathbf{A}(t-\theta)} \sum_i \mathbf{B}_i \mathbf{u}(\theta - \alpha_i)\, d\theta \quad \text{(C-17)}
$$

given $\mathbf{x}_0$ the vector of initial conditions for $t = 0$ or in general, for an arbitrary starting time t

$$
\mathbf{x}(t + \tau) = e^{\mathbf{A}\tau}\mathbf{x}(t) + \int_t^{t+\tau} e^{\mathbf{A}(t+\tau-\theta)} \sum_i \mathbf{B}_i \mathbf{u}(\theta - \alpha_i)\, d\theta \quad \text{(C-18)}
$$

From (C-18) for any k,

$$
\mathbf{x}(t + \alpha_k) = e^{\mathbf{A}\alpha_k}\mathbf{x}(t) + \int_t^{t+\alpha_k} e^{\mathbf{A}(t+\alpha_k-\theta)} \sum_i \mathbf{B}_i \mathbf{u}(\theta - \alpha_i)\, d\theta \quad \text{(C-19)}
$$

Expanding out the inner summation over i and recalling the structure of the $\mathbf{B}_i$ matrices, a term by term evaluation for each member of the vector $\mathbf{x}$ shows that, in particular

$$
x_k(t + \alpha_k) = e^{a_k\alpha_k} x_k(t) + \int_t^{t+\alpha_k} e^{\mathbf{A}(t+\alpha_k-\theta)}\, \mathbf{B}_k \mathbf{u}(\theta - \alpha_k)\, d\theta \quad \text{(C-20)}
$$

where $\mathbf{B}_k$ is the only contributor to x_k, for all k through its single non-zero element b_k. If we then proceed to define a diagonal matrix

* When some poles are repeated, some simple approximations are required to obtain the form in (C-12) (see Ogunnaike, in press).

$$P = \begin{bmatrix} e^{a_1\alpha_1} & & & & 0 \\ & e^{a_2\alpha_2} & & & \\ & & \cdot & & \\ & & & \cdot & \\ 0 & & & & e^{a_n\alpha_n} \end{bmatrix}$$

and recall Equation (C-16) for $x^*(t)$, it follows directly from (C-18) that

$$x^*(t) = Px(t) + \int_t^{t+\alpha_1} e^{A(t+\alpha_1-\theta)} B_1 u(\theta - \alpha_1)d\theta + \int_t^{t+\alpha_2} e^{A(t+\alpha_2-\theta)} B_2 u(\theta - \alpha_2)d\theta + \cdots + \int_t^{t+\alpha_n} e^{A(t+\alpha_n-\theta)} B_n u(\theta - \alpha_n)d\theta$$

or

$$x^*(t) = Px(t) + \int_{-\alpha_1}^0 e^{A(-\theta)}B_1 u(t+\theta)d\theta + \int_{-\alpha_2}^0 e^{A(-\theta)}B_2 u(t+\theta) \cdot d\theta + \ldots + \int_{-\alpha_n}^0 e^{A(-\theta)} B_n u(t+\theta) \cdot d\theta \quad \text{(C-21)}$$

Recognizing α_n to be the largest delay, (C-21) could be rewritten as

$$x^*(t) = Px(t) + \int_{-\alpha_n}^0 E(\rho, \theta)u(t+\theta) \cdot d\theta \quad \text{(C-22)}$$

where $E(\rho, \theta)$ is defined by

$$\left.\begin{aligned} E(\alpha_n, \theta) &= e^{A(-\theta)} B_n \\ E(\alpha_j^+, \theta) &= E(\alpha_j^-, \theta) + e^{A(-\theta)} B_j \end{aligned}\right\} \quad \text{(C-23)}$$

$$j = n-1,\ n-2,\ \ldots .\ 2, 1$$

From (C-22)

$$y^* = C'x^* = C'Px(t) + C'\int_{-\alpha_n}^0 E(\rho, \theta)u(t+\theta) \cdot d\theta \quad \text{(C-24)}$$

The feedback law is

$$u(t) = \begin{cases} -G_c y^* & \text{(Load disturbances)} \\ G_c(y_d - y^*) & \text{(Set-point changes)} \end{cases}$$

where G_c is the appropriately chosen matrix of controller gains. y_d is the desired set-point. From (C-24) then

$$\boxed{u(t) = G_c y_d - G_c\left\{C'Px(t) + C'\int_{-\alpha_n}^0 E(\rho, \theta)u(t+\theta)d\theta\right\}} \quad \text{(C-25)}$$

where, for load disturbances, $y_d = 0$. The optimal control law derived by Soliman and Ray (1972) for such linear systems represented by (C-15) is

$$u(t) = -W^{-1}\left\{E_3^T(t, 0)x(t) + \int_{-\beta_b}^0 E_4(t, 0, s)u(t+s) \cdot ds\right\} \quad \text{(C-26)}$$

where the parameters are determined from the solution of the ordinary and partial differential Ricatti equations shown in detail in Soliman and Ray (1972). The important detail to note are the discontinuities in E_4 given as

$$\left.\begin{aligned} E_4(t, -\beta_b, s) &= B_b^T E_3(t, s) \\ E_4(t, -\beta_j^+, s) &= E_4(t, -\beta_j^-, s) + B_j^T E_3(t, s); \end{aligned}\right\} \quad \text{(C-27)}$$

$$j = 1, 2, \ldots b-1$$

where $\beta_j;\ j = 1, 2, \ldots .\ b$ are the time delays and β_b = maximum delay.

By inspecting (C-25) with (C-23), and (C-26) with (C-27), it is easy to see that, apart from the optimal choice of parameters, (C-25) and (C-26) have the same structure. The quantity C' occurs in (C-25) because we here consider the control of $y = C'x$, while Soliman and Ray considered the control of x.

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Heat Transfer to a Stationary Sphere in a Plasma Flame

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The overall heat transfer rates to stationary spheres of highly-polished molybdenum (2 to 5.6 mm in diameter) in a confined argon plasma jet were experimentally measured. The effects of large temperature differences and large variations of the fluid property on the heat transfer process are investigated. The sphere Reynolds number range between 10 and 80, and the sphere temperature between 1,200 K and 2,400 K, with temperature differences between the gas and the sphere in excess of 2,000 K. These conditions are within the range commonly encountered in industrial plasma processing. The experimental values of the heat transfer coefficients are in reasonably good agreement with the predictions of a theoretical analysis of variable-property flow and heat transfer previously reported by the authors.

SCOPE

This study is part of a continuing program of investigation by the Plasma Technology Group to develop applications of plasmas to chemical and metallurgical processes of industrial interest. The availability of reliable plasma generating devices of industrial size (up to 10 MW) with a great deal of design flexibility is currently prompting a very active search to take advantage of some unique features that can lead to truly novel processes. Plasma systems offer continuous operation, ease of control, good energy utilization, and not the least important, an enormous concentration of energy in a very small volume, resulting in very high temperature levels and rates of throughput at or near atmospheric pressure. Heterogeneous solids-gas reactions involving the contacting of fine particles with an entraining plasma gas appear to be particularly promising in this regard.

The separation of refractory metals of high unit value (molybdenum, zirconium, tungsten, titanium, etc.) from concentrates of their ores; the production of valuable oxides (zirconia, TiO_2, ultra-pure silica); the production of ferroalloys (ferromolybdenum, ferrocolumbium, ferrovanadium); and the production of acetylene from coal are all examples of processes now under active investigation. However, due to the extremely short residence time of the particles in the high enthalpy plasma stream (typically, a few millisec), the design of a plasma reactor largely depends upon an accurate knowledge of the fluid mechanics and heat transfer characteristics of the system. The latter, in turn, enables the proper control of the thermal histories of the particles and consequently, of their degree of conversion. Comprehensive models to predict the temperature history of particles entrained in a plasma were presented by Boulos and Gauvin (1974) for a one-dimensional system, and by Bhattacharyya and Gauvin (1975) for a three-dimensional jet, using the thermal decomposition of MoS_2 into metallic molybdenum particles and elemental sulphur as an example in these simulations. Both studies emphasized the importance of heat transfer considerations in effecting the desired conversion.

Heat transfer to particles entrained by a plasma exhibits some unusual characteristics. First of all, the particle Reynolds number is remarkably low, typically less than 50, owing to the unusually high value of the plasma's kinematic viscosity. More important still, is the fact that in most heterogeneous plasma systems, the temperature of the particle may range from 1,500 to 2,500 K, while that of the surrounding plasma extends from 5,000 K upward. Under these conditions of very large temperature differences, it is no longer permissible to assume that the fluid immediately surrounding the particle exhibits constant properties, as is commonly done in more conventional situations at lower temperatures.

A theoretical study was recently published (Sayegh and Gauvin 1979) in which the effects of large temperature differences on the rate of pure heat transfer to a stationary

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